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AN AXIOMATICS FOR BICATEGORIES OF MODULES

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Introduction

The main result of this paper is that the construction, from a bicategory **B** (with stable local colimits), of the bicategory **B-mod**, of **B**-categories and **B**-modules, is *idempotent*. This generalises the basic fact of sheaf theory that the process of forming sheaves on a site is idempotent (which fact is the origin of the notion of Grothendieck topos).

This construction **B-mod** which generalises the construction of rings and modules from abelian groups, and at the same time the construction of sheaves from a site, has been investigated in [3–5, 9, 18, 19] from the point of view of examples. The idempotence theorem provides, in addition, a powerful formal motivation for the study of **B-mod**. To express the fact that **(V-mod)mod** is biequivalent to **V-mod** (**V** a cocomplete *monoidal category*), it is necessary to define categories enriched over the *bicategory* **V-mod**.

An immediate consequence of the idempotence theorem is a precise characterization of bicategories of the form **B-mod** as those bicategories with stable local colimits for which each 'category is representable as an object' (this is the content of our notion of *collage of a category*). There is already a characterization of **B-mod** due to Street [16], phrased in terms of lax colimits, which does not proceed from the idempotence theorem and which is less precise in that it assumes also the existence of 'generators'. We deduce Street's theorem from ours in Section 6.

Bicategories with stable local colimits are analogous to additive categories (and so to rings); it is well known that the addition of arrows can be derived from the fact that 'products = coproducts'. A result of Wood [21] allows us to obtain a similar characterization of bicategories of modules as bicategories for which 'lax limits = lax colimits' (see Section 3 for a precise statement).

Other aspects of the paper are

(i) we introduce in Section 1, and use extensively, a 'calculus of modules' due

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essentially to Wood [20], but further developed here;

(ii) in Section 4 we break the construction **B-mod** into two steps which are interesting in their own right: the construction of monads **Mon(B)**, and the construction of matrices **Matr(B)** from a bicategory **B**.

Both of those processes are idempotent and, in fact, the idempotence of the module construction could have been alternatively deduced from the fact that **B-mod** is biequivalent to **Mon(Matr(B))**.

The standard references to bicategory theory are [1, 11, 14].

1. Notation and preliminaries

In this section we recall briefly some standard notions of bicategory theory and establish our notation. We also describe a calculus for modules.

We assume the reader is acquainted with the notions of bicategory, lax functor (or morphism) and homomorphism of bicategories, lax transformations and optransformations of lax functors, modifications, and equivalent objects in a bicategory, as defined, for example, in [1, 14]. Following [14], we call **HOM(B,C)** the bicategories of homomorphisms (strong transformations and modifications) between two bicategories **B** and **C** and **Bicat(B,C)** the bicategory of lax functors (lax transformations and modifications) between them.

A *distributive bicategory* **B** is a locally cocomplete bicategory (that is, $\mathbf{B}(u,v)$ is small cocomplete for each pair u, v of objects of **B**) with colimits preserved by composition on both sides. In the present paper bicategories are assumed to be distributive, except in Section 3, where we describe properties which *imply* distributivity.

As for the notions of **B**-categories (that is, categories enriched over **B**), **B**-functors and **B**-modules we refer to [4, 15, 18], limiting ourselves here to recalling some terminology and indicating some notation. The conventions adopted in the papers mentioned above do not always agree: we will follow here the conventions of [18]. If X is a **B**-category and x an object of X, the 'underlying object' of x is denoted by ex and the hom between x and y (y in X) by $X(x,y): ex \to ey$. A **B**-module $R: X \,+\!\!\!\!\to Y$, evaluated at x in X and y in Y, is denoted $R(x,y)$ and our convention is that R is contravariant in x and covariant in y. The bicategory of small **B**-categories (that is, with a small set of objects) and modules is denoted by **B-mod**.

A module with right adjoint is called a *map* (as is any arrow with a right adjoint in a bicategory); the bicategory of **B**-categories and maps is denoted by **Map(B-mod)**. Let us recall also the embedding $(\,)\hat{}: \mathbf{B} \to \mathbf{B}\text{-}\mathbf{mod}$ which assigns to the object u the **B**-category $\hat{u}$ with only one object (say u again) and with $\hat{u}(u,u) = 1_u$ and to each arrow $s: u \to v$ the module $\hat{s}(u,v) = s$. Often we do not distinguish notationally between $\hat{u}$ and u.

Finally, let us mention the fact that we will use, quite informally, the 'bar' notation, commonly used for denoting bijections, here to denote also equivalences (see, e.g. the proof of Proposition 3.2).

1.1. The calculus of modules

We describe below a calculus which has its genesis in [17], was developed in [20], is further developed here, and which we hope will make calculations with modules quite transparent.

Notation 1.1. Given module $R: Z \to W$ and functors $f: X \to Z$, $g: Y \to W$,

$$
\begin{tikzcd}
Z \arrow[r, "R", "+" marking] & W \\
X \arrow[u, "f"] \arrow[r, "R(f,g)"', "+" marking] & Y \arrow[u, "g"']
\end{tikzcd}
$$

denote by $R(f,g): X \nrightarrow Y$ the module with
- components

$$R(f,g)(x,y) = R(fx, gy),$$

- actions

$$Y(y,y')R(fx,gy) \xrightarrow{\text{effect of } g \cdot 1} W(gy,gy')R(fx,gy) \xrightarrow{\text{action}} R(fx,gy'),$$

$$R(fx,gy)X(x',x) \xrightarrow{1 \cdot \text{effect of } f} R(fx,gy)Z(fx',fx) \xrightarrow{\text{action}} R(fx',gy).$$

Examples 1.2. (i) If $z: u \to Z$ and $w: v \to W$ are objects of Z and W, then $R(z,w)$ is the component of R at (z,w).

(ii) Denoting the identity module on X simply by X, notice that a functor $h: X \to Y$ induces modules $Y(h,1): X \nrightarrow Y$ and $Y(1,h): Y \nrightarrow X$.

We list some standard properties of modules using this notation.

Properties 1.3. (i) $Y(h,1)$ is left adjoint to $Y(1,h)$. Denote the unit by η_h, the counit by ε_h. (In agreement with the notation introduced earlier for maps, the modules $Y(h,1)$ and $Y(1,h)$ are often denoted h_* and h^* respectively.)

(ii) $Z(k,1)Y(h,1) \cong Z(kh,1)$, $Y(1,h)Z(1,k) \cong Z(1,kh)$ for any functors $h: X \to Y$, $k: Y \to Z$.

(iii) (Yoneda lemma) $Y(1,k)Y(h,1) \cong Y(h,k)$.

(iv) $R(f,g) \cong W(1,g) \cdot R \cdot Z(f,1)$.

In terms of the data given above we can describe the important features of modules and functors.

Example 1.4 (Composition in a category). If x, y, z are objects of X, the composition

$X(y,z)\cdot X(x,y)\to X(x,z)$ is the 2-cell

$$X(1,z)X(y,1)X(1,y)X(x,1)\xrightarrow{1\cdot\varepsilon_y\cdot 1}X(1,z)X(x,1).$$

Example 1.5 (Effect of a functor). If $f:X\to Y$ is a functor and $x,x'\in X$, then the effect of f is

$$X(x,x')\cong X(1,x')X(x,1)\xrightarrow{1\cdot\eta_f\cdot 1}X(1,x')Y(1,f)Y(f,1)X(x,1)\cong Y(fx,fx').$$

Example 1.6 (Action of a module). If $R:X\to\!\!\!\!\!+\ Y$ is a module and $x,x'\in X$, $y\in Y$, then the action of X on R

$$R(x,y)\cdot X(x',x)\to R(x',y)$$

is given by

$$Y(1,y)\cdot R\cdot X(x,1)X(1,x)X(x',1)\xrightarrow{1\cdot\varepsilon_x\cdot 1}Y(1,y)\cdot R\cdot X(x',1).$$

Example 1.7 (Composition of modules). If $R:X\to\!\!\!\!\!+\ Y$, $S:Y\to\!\!\!\!\!+\ Z$ and $T:X\to\!\!\!\!\!+\ Z$ are modules, then a 2-cell $\alpha:S\cdot R\to T$ is the same as a family of 2-cells $\alpha_y:S\cdot Y(y,1)Y(1,y)\cdot R\to T$, indexed by the objects y of Y, natural in T, and satisfying

$$\alpha_y(1\cdot\varepsilon_{y'}\cdot 1)=\alpha_{y'}(1\cdot\varepsilon_y\cdot 1):S\cdot Y(y,1)Y(1,y)Y(y',1)Y(1,y')\cdot R\to T;$$

explicitly, the bijection is

$$(\alpha:S\cdot R\to T)\leftrightarrow(\alpha\cdot(1\cdot\varepsilon_y\cdot 1))_{y\in Y}.$$

2. Idempotence and characterization theorems

Definition. Let **B** be a distributive bicategory. A *collage* of a **B**-category X is an object $\operatorname{coll}X$ of **B** such that $(\operatorname{coll}X)\hat{}$ is equivalent to X in **B-mod**.

Remark 2.1. A distributive bicategory **B** admits collages of all small **B**-categories iff the homomorphism $(\,)\hat{}:\mathbf{B}\to\mathbf{B}\text{-}\mathbf{mod}$ is a biequivalence: it is clear that $(\,)\hat{}$ is locally an equivalence of categories; to say that collages exist is just to say that $(\,)\hat{}$ is essentially surjective.

Proposition 2.2 (Idempotence theorem). *Let* **B** *be a distributive bicategory. The bicategory* **B-mod** *admits collages of all small categories, and hence* **B-mod** *is biequivalent to* **(B-mod)-mod**.

Proof. Given a **B-mod**-category X, we construct a **B**-category $W=\operatorname{coll}X$ as follows: the set of objects of W is the disjoint union $\coprod_{x\in X}ex$ of the **B**-categories ex and, if $\xi\in ex$, $\upsilon\in ey$, $W(\xi,\upsilon)=X(x,y)(\xi,\upsilon)$. The composition is obtained by first observing that if $\zeta \in ez$, then

$$W(\upsilon,\zeta)W(\xi,\upsilon) \cong ez(1,\zeta)X(1,z)X(y,1)ey(\upsilon,1)ey(1,\upsilon)X(1,y)X(x,1)ex(\xi,1).$$

Then, composition is the 2-cell obtained by first taking

$$\varepsilon_\upsilon : ey(\upsilon,1)ey(1,\upsilon) \to ey$$

and then

$$\varepsilon_y : X(y,1)X(1,y) \to X.$$

Now, for each x, consider the **B**-functor $i_x : ex \to W$ defined as follows: $i_x(\xi)=\xi$ ($\xi \in ex$) and the effect of i_x is given by

$$\eta_x(\xi,\xi') : ex(\xi,\xi') \to W(i_x\xi, i_x\xi'), \quad \text{where } \eta_x : ex \to X(1,x)X(x,1) \cong X(x,x).$$

We claim that

$$W(i_x,i_y) \cong X(x,y) : ex \to ey. \tag{1}$$

Clearly $W(i_x,i_y)(\xi,\upsilon) = W(\xi,\upsilon) = X(x,y)(\xi,\upsilon)$. We need to compare the actions: let us look at the action of ex on $W(i_x,i_y)$, given by

$$\begin{aligned}
&W(i_x,i_y)(\xi,\upsilon)ex(\xi',\xi) \xrightarrow{1\cdot\text{effect of } i_x} W(i_x,i_y)(\xi,\upsilon)W(i_x,i_x)(\xi',\xi)\\
&\xrightarrow{\text{composition in } W} W(i_x,i_y)(\xi',\xi)\\
&\cong X(x,y)(\xi,\upsilon)ex(\xi',\xi) \xrightarrow{1\cdot\eta_x(\xi',\xi)} X(x,y)(\xi,\upsilon)X(x,x)(\xi',\xi)\\
&\xrightarrow{1\cdot\varepsilon_\xi\cdot 1} X(x,y)X(x,x)(\xi',\xi) \xrightarrow{1\cdot\varepsilon_x\cdot 1} X(x,y)(\xi',\upsilon)\\
&\cong X(x,y)(\xi,\upsilon)ex(\xi',\xi) \xrightarrow{1\cdot\varepsilon_\xi\cdot 1} X(x,y)(\xi',\upsilon)\\
&\quad\text{(triangular identity on } \eta_x,\varepsilon_x)\\
&\cong \text{action of } ex \text{ on } X(x,y).
\end{aligned}$$

Now, to show that W is the collage of X, we define modules

$$R : X \nrightarrow W \quad\text{and}\quad S : W \nrightarrow X$$

as follows: the components are

$$R(x) = W(i_x,1) : ex \to W, \qquad S(x) = W(1,i_x) : W \to ex.$$

The action of X on R is defined by the composite

$$\begin{aligned}
&ey(1,\zeta)W(1,i_y)R(x)X(x',x)ex'(\xi',1) \cong ey(1,\zeta)W(i_x,i_y)X(x',x)ex'(\xi',1)\\
&\cong ey(1,\zeta)X(x,y)X(x',x)(\xi',1) \xrightarrow{1\cdot\varepsilon_x\cdot 1} X(x',y)(\xi',\zeta).
\end{aligned}$$

Directly from the definition of this action we obtain that

$$(W(1,i_x))\hat{}\cdot R : X \to ex \cong X(1,x), \tag{2}$$

and similarly

$$S\cdot (W(i_x,1))\hat{} \cong X(x,1).$$

Finally, to check that R and S are inverse equivalences, notice that

$$S(y)\cdot R(x) = W(1,i_y)W(i_x,1) \cong W(i_x,i_y) \cong X(x,y) \quad \text{(by (1))},$$

and

$$\begin{aligned} W(1,\upsilon)\hat{}\cdot R\cdot S\cdot W(\xi,1)\hat{} &\cong ey(1,\upsilon)\hat{}\,W(1,i_y)\hat{}\cdot R\cdot S\cdot W(i_x,1)\hat{}\,ex(\xi,1)\hat{} \\ &\cong ey(1,\upsilon)\hat{}\,X(1,y)X(x,1)ex(\xi,1)\hat{} \cong ey(1,\upsilon)\hat{}\,X(x,y)ex(\xi,1)\hat{} \quad \text{(by (2))} \\ &\cong W(\xi,\upsilon). \end{aligned}$$

It is straightforward to check that the actions of $R\cdot S$ and W agree. □

Remark 2.3. A **B**-category X can be thought of as a **B-mod**-category via the embedding $(\,)\hat{}$. The collage of X in **B-mod** is then X seen as an *object* of **B-mod**.

Corollary 2.4 (Characterization theorem). *Let* **B** *be a distributive bicategory. Then* **B** *is biequivalent to* **W-mod**, *for some distributive bicategory* **W**, *iff* **B** *admits collages of small categories.*

We next show that the property of collage of a category is the same as an apparently weaker *left* universal property.

Proposition 2.5. *An object v of* **B** *is the collage of* **B**-*category X iff there is a module $R : X \,\mathrel{-\!\!\!+\!\!\!\to}\, v$ inducing (by composition) an equivalence of categories*

$$\mathbf{B\text{-}mod}(X,\hat{w}) \simeq \mathbf{B}(v,w) \quad \textit{for each } w \textit{ in } \mathbf{B}.$$

Proof. Suppose $R : X \to \hat{v}$ satisfies the left universal property described in the statement of the proposition. We will construct a module $S : \hat{v} \,\mathrel{-\!\!\!+\!\!\!\to}\, X$ such that $S\cdot R \cong 1_X$ and $R\cdot S \cong 1_{\hat{v}}$. By the universal property, we have a family of 1-cells $S(x)$ in **B** corresponding to the representables

$$\frac{X(1,x) : X \,\mathrel{-\!\!\!+\!\!\!\to}\, (ex)\hat{}\,;\ x\in X}{S(x) : v \to ex}$$

and there is an action of X on $S(x)$, $x\in X$, exhibiting S as a module as follows. For all $y\in X$, we have

$$X(x,y)\cdot S(x)\cdot R \overset{\cong}{\Rightarrow} X(x,y)X(1,x) \to X(1,y) \overset{\cong}{\Rightarrow} S(y)\cdot R.$$

So, from the universal property, we get $X(x,y)\cdot S(x) \to S(y)$. It is clear that $S\cdot R \cong 1_X$. Further, observe that $R\cdot S\cdot R \cong R\cdot 1_X \cong R$ and this, again by the universal property, induces an isomorphism $R \cdot S \cong 1_{\hat{v}}$. The converse follows from the fact that $(\ )^{\hat{}}$ is locally an equivalence of categories. □

Remarks 2.6. (i) Of course this result gives a strengthening of the characterization theorem: to see whether a bicategory is of the form **W-mod** we need only to show the existence, for each category X, of an object satisfying the *left* universal property of collage.

(ii) By duality, the notion of collage of a category is also the same as the following *right* universal property: an object v is the collage of a **B**-category X if there is a module $S : \hat{v} \relbar\!\!\!\mapsto X$ inducing an equivalence of categories

$$\mathbf{B}\text{-}\mathbf{mod}(\hat{w}, X) \simeq \mathbf{B}(w, v) \quad \text{for each } w \text{ in } \mathbf{B}.$$

We call the components of S the *projections* of the collage, and the components of R (the inverse equivalence of S) the *coprojections* of the collage.

Proposition 2.7. *In a distributive bicategory which admits collages of small categories we have*

(i) *The coprojections of a collage are maps; the projections are right adjoints to the coprojections;*

(ii) *If modules $R : X \relbar\!\!\!\mapsto \hat{v}$ and $S : \hat{v} \relbar\!\!\!\mapsto X$ are inverse equivalences presenting v as the collage of X, then the actions of S are the* mates *(in the sense of* [11, p. 87]) *of the actions of R;*

(iii) *An arrow with as domain a collage is a map iff its composition with each coprojection is a map; an arrow with as codomain a collage is a right adjoint iff its composition with each projection is a right adjoint.*

Proof. (i) Follows from the fact that the coprojections of the collage of X are of the form $R \cdot X(x, 1)$, and the projections are of the form $X(1, x) \cdot S$, where R and S are inverse equivalences.

(ii) Is a straightforward application of the calculus of modules.

(iii) Follows from the general fact that a module $T : X \relbar\!\!\!\mapsto \hat{w}$ has a right adjoint if (and only if) each of the components T_x has a right adjoint. □

3. Lax colimits

In this section we develop a different characterization of bicategories of modules, in which there is no assumption of local colimits (just as additive categories can be characterized without using the local addition).

First we need the notion of *collage of a lax functor* (or lax colimit) and the dual notion of *opcollage* (or lax limit).

Definition 3.1 (Street [16]). Given bicategories **D** and **B** (not necessarily distributive) and a lax functor $L: \mathbf{D} \to \mathbf{B}$, a *collage* of L is an object l of $\mathbf{B}$ together with an op-transformation $\lambda: L \to \ulcorner l \urcorner$ (the constant homomorphism $\mathbf{D} \to \mathbf{B}$ with $\ulcorner l \urcorner(d) = l$, for all objects d of $\mathbf{D}$) inducing an equivalence of categories

$$\mathbf{B}(l, w) \simeq \mathbf{Bicat}(\mathbf{D}^{\mathrm{op}}, \mathbf{B}^{\mathrm{op}})(L, \ulcorner w \urcorner).$$

The dual definition is obvious; we will sometimes call the right-hand side **Lax-cones**(L, w) (respectively **Lax-cones**(w, L)).

It is clear that the *collage of a* $\mathbf{B}$*-category* X, as defined at the beginning of Section 2, is a particular case of *collage of a lax functor* (a $\mathbf{B}$-category is a lax functor from the chaotic category on the set X into $\mathbf{B}$).

Proposition 3.2. *A distributive bicategory* $\mathbf{B}$ *admits collages of small categories iff it admits collages of lax functors with small domain* (*in the sense of the definition above*).

Proof. Given a lax functor $L: \mathbf{D} \to \mathbf{B}$, we will construct a $\mathbf{B}$-category $\tilde{L}$ such that

$$\mathbf{B}\text{-}\mathbf{mod}(\tilde{L}, \hat{w}) \simeq \mathbf{Bicat}(\mathbf{D}^{\mathrm{op}}, \mathbf{B}^{\mathrm{op}})(L, \ulcorner w \urcorner).$$

The $\mathbf{B}$-category $\tilde{L}$ has the same objects as $\mathbf{D}$ and $ed = Ld$ (for all objects d in $\mathbf{D}$). Further, $\tilde{L}(d, d') = \operatorname{colim} L_{dd'}: \mathbf{D}(d, d') \to \mathbf{B}(Ld, Ld')$. Given a module $R: \tilde{L} \to \hat{w}$, the components λ_d of the corresponding optransformation $\lambda: L \to \ulcorner w \urcorner$ are given by $\lambda_d = R_d$. As for the 2-cells involved, observe that those of the action of the module R correspond with the 2-cells of the optransformation by the following bijection:

$$\begin{array}{cl}
R_{d'} \cdot \tilde{L}(d, d') \to R_d & \\
\hline
R_{d'} \cdot \operatorname{colim} L_{dd'} \to R_d & \text{(by definition)} \\
\hline
\operatorname{colim}(R_{d'} \cdot L_{dd'}) \to R_d & \text{(colims are preserved by composition)} \\
\hline
R_{d'} \cdot a \to R_d,\ a \in L_{dd'} & \\
\hline
\lambda_{d'} \cdot a \to \lambda_d,\ a \in L_{dd'} & \text{(cone)}
\end{array}$$

Moreover, 2-cells between modules and modifications between optransformations correspond bijectively (being both determined by the same family of 2-cells in $\mathbf{B}$ between the components). So, if $\mathbf{B}$ has *collages of small categories*, then

$$\mathbf{Bicat}(\mathbf{D}^{\mathrm{op}}, \mathbf{B}^{\mathrm{op}})(L, \ulcorner w \urcorner) \simeq \mathbf{B}\text{-}\mathbf{mod}(\tilde{L}, \hat{w}) \simeq \mathbf{B}(\operatorname{coll} \tilde{L}, w),$$

i.e. $\mathbf{B}$ has *collages for any morphism* $L: \mathbf{D} \to \mathbf{B}$. $\square$

It is clear that Proposition 2.7 for collages of categories implies a similar result for collages of lax functors. In fact bicategories of modules can be characterized as follows, without the assumption of distributivity.

Proposition 3.3 (Characterization theorem). *A bicategory* $\mathbf{B}$ *is biequivalent to* $\mathbf{W}$**-mod** *for some distributive bicategory* $\mathbf{W}$ *iff*

(i) **B** *admits collages of lax functors with small domain, and the coprojections are maps;*

(ii) *If lax cone $\varrho : L \to v$ exhibits v as the collage of L, then the following describes a lax cone $\sigma : v \to L$ exhibiting v as opcollage of L: projections of σ are right adjoints of the coprojections of ϱ; 2-cells of cone σ are* mates [11] *of the 2-cells of ϱ;*

(iii) *An arrow out of a collage is a left adjoint iff each composite with a coprojection is a left adjoint; an arrow into an opcollage is a right adjoint iff each composite with a projection is a right adjoint.*

Proof. It clearly suffices to show that **B** is distributive: a proof of this can be found in [21]. We here give a new uniform construction of the local colimits in $\mathbf{B}(u, v)$, where u and v are objects of **B**. A diagram $F : C^{\mathrm{op}} \to \mathbf{B}(u, v)$ induces an oplax transformation τ from $\ulcorner u \urcorner : C \to \mathbf{B}$ to $\ulcorner v \urcorner : C \to \mathbf{B}$ with $\tau_c = Fc$ and $\tau_\alpha = F\alpha$ ($\ulcorner u \urcorner$ and $\ulcorner v \urcorner$ are the constant homomorphisms at $\ulcorner u \urcorner$, $\ulcorner v \urcorner$ respectively). This in turn induces an arrow $\mathrm{coll}(\ulcorner u \urcorner) \to \mathrm{coll}(\ulcorner v \urcorner)$. Now the trivial cone $u \to \ulcorner u \urcorner$ induces a right adjoint arrow $\Delta : u \to \mathrm{coll}(\ulcorner u \urcorner)$ and the trivial cone $\ulcorner v \urcorner \to v$ induces a left adjoint arrow $\nabla : \mathrm{coll}(\ulcorner v \urcorner) \to v$. Then

$$\operatorname{colim} F = u \xrightarrow{\Delta} \mathrm{coll}(\ulcorner u \urcorner) \to \mathrm{coll}(\ulcorner v \urcorner) \xrightarrow{\nabla} v.$$

We omit the check of the colimit property and the fact that these colimits are preserved by composition. $\square$

4. Monads, matrices and examples

In this section we will describe some special cases with examples which are interesting in their own right and familiar in other contexts. Further, in these cases the assumptions on the bicategory **B** can be significantly refined, so we will review the definitions with the restricted assumptions.

4.1. The bicategory of monads

Let **B** be a bicategory with local coequalizers stable under composition (for example a distributive bicategory) and u an object of **B**.

Recall that a *monad u_m on u in* **B** is a monoid in the monoidal category $\mathbf{B}(u, u)$; that is a 1-cell $m : u \to u$ and two 2-cells

$$\eta_m : 1_u \to m, \qquad \mu_m : mm \to m$$

satisfying the usual associativity and identity equations.

A *bimodule* $R : u_m \nrightarrow v_n$ from the monad m to the monad n is a 1-cell $R : u \to v$ together with two actions

$$l_R : Rm \to R, \qquad r_R : nR \to R$$

satisfying besides the usual (left and right) unit and associative laws, also the equation expressing the commutativity of the two actions.

Given bimodules $R : u_m \to\!\!\!\!\!\!+\ v_n$ and $S : v_n \to\!\!\!\!\!\!+\ z_p$ we define their *composite* $S \otimes_n R : u_m \to\!\!\!\!\!\!+\ z_p$ by

$$S \otimes_n R = \mathrm{coeq}\left(S \cdot n \cdot R \underset{l_S \cdot R}{\overset{S \cdot r_R}{\rightrightarrows}} S \cdot R \right).$$

Further, given bimodules R and S with common domain and codomain, a *morphism of bimodules* is a 2-cell from R to S satisfying compatibility conditions with respect to the actions. Let us denote by **Mon(B)** the *bicategory of monads* in **B**, whose objects are the same as **B**, 1-cells are bimodules, 2-cells are morphisms of bimodules; composition is bimodule composition and the identity 1-cell of u_m is just m itself. Notice that **Mon(B)** also has local coequalizers stable under composition.

When **B** is distributive, **Mon(B)** is exactly the subbicategory of **B-mod** consisting of one-object categories. As a particular case of Proposition 2.2, we know that one-object categories *over* **Mon(B)** have collages, which are objects of **B-mod**, i.e. **B**-categories. Examination of the construction reveals that collages of one-object categories are *actually in* **Mon(B)**. Further, if we restrict to one-object categories, the proof of Proposition 2.2 requires only the existence of stable coequalizers in **B**.

If **B** has local coequalizers, we define the *collage of a monad* u_m to be an object of **B** such that the identity monad over it is equivalent to u_m in **Mon(B)**.

Proposition 4.1 (Idempotence theorem for **Mon(B)**). *If* **B** *is a bicategory with local coequalizers stable under composition, the bicategory* **Mon(B)** *admits collages of monads, and hence* **Mon(B)** *is biequivalent to* **Mon(Mon(B))**.

Corollary 4.2 (Characterization theorem for **Mon(B)**). *Let* **B** *be a bicategory with local coequalizers stable under composition. Then the following conditions are equivalent:*

(i) **B** *admits collages of monads;*

(ii) **B** *is biequivalent to* **Mon(W)**, *for some bicategory* **W** *with local stable coequalizers.*

Notice that, due to the self-duality of the characterization above, if **B** is a bicategory of monads, then also $\mathbf{B}^{\mathrm{op}}$ is.

It is evident that the left universal property of the collage of a monad is the universal property defining the *Kleisli* object for a monad (see [16]; collages correspond then to Kleisli objects which are also *Eilenberg-Moore*). We call a bicategory *Kleisli-complete* if it admits the Kleisli construction for every monad. We are led by Proposition 2.5 to a characterization theorem in terms of Kleisli constructions.

Corollary 4.3. *A bicategory* **B** *with local stable coequalizers is biequivalent to* **Mon(W)** *for some bicategory* **W** *with local stable coequalizers iff it is Kleisli-complete.*

Example 4.4. Let **V** be a monoidal category, seen as a one-object bicategory, with coequalizers stable under composition (for example a cocomplete monoidal closed category). Then **Mon(V)** is the bicategory of monoids in **V** and bimodules between them. In the classical case $\mathbf{V}=\mathbf{AB}$ we obtain of course as **Mon(V)** the bicategory of rings and bimodules, composition being the usual composition of bimodules.

Example 4.5. Let **LTOP** denote the bicategory with objects toposes and hom-categories given by $\mathbf{LTOP}(X,Y)=\mathbf{LEX}(X,Y)^{\mathrm{op}}$. By a theorem of Lawvere and Tierney, **LTOP** is then Kleisli-complete: a monad in **LTOP** is in fact a left exact comonad on a topos, the Kleisli construction yields the topos of coalgebras and its universal 1-cell is the cofree functor, whose right adjoint (the forgetful functor) exhibits the topos of coalgebras as an Eilenberg–Moore object. This observation is essentially contained in [21], while the whole setup of the example appears in [13].

Example 4.6. Let **B** be **Span**(E), where E is a left exact category with pullback-stable coequalizers. Then **Mon(B)** is the bicategory **Prof**(E) of internal categories and internal profunctors, and admits collages of monads.

Example 4.7. Let **B** be the bicategory **Rel**(E), where E is a regular category. Then **Mon(B)** is the bicategory **Ord**(E) of ordered objects and ideals (see [7]). Of course **B** is not Kleisli-complete, but if moreover E is exact (in the sense of Barr), then *symmetric* monads (i.e. equivalence relations $e:u\to u$) have as collages just the quotients u/e, i.e. the two objects u_e and $1_{u/e}$ are equivalent in **Ord**(E). With this said, if we call **Eq**(E) the subbicategory of **Ord**(E) given by the equivalence relations, we get a biequivalence $\mathbf{B}\simeq\mathbf{Eq}(E)$, which characterizes bicategories of relations on exact categories among those on regular categories.

4.2. *The bicategory of matrices*

Let V be a monoidal category with small coproducts preserved by the tensor product. We define a bicategory **Matr(V)** of **V**-matrices as follows: the objects are small sets and 1-cells $r:X\to Y$ are families $(r_{ij})_{i,j\in X\times Y}$ of objects of **V**; 2-cells are defined 'pointwise', i.e. a 2-cell $\alpha:r\to s$ is a family $(r_{ij}\to s_{ij})_{i,j\in X\times Y}$ of arrows in **V**. Composition of 1-cells is just 'matrix multiplication'.

Notice that **Matr(V)** is a bicategory with local small coproducts, stable under composition. Therefore, in order to iterate the construction, we must get a generalization of it to bicategories with local small coproducts. This has been done in full detail in [4], so we just recall briefly the definition.

Let **B** be a bicategory with local small coproducts, stable under composition. Then **Matr(B)** is defined as follows: the objects are small families of objects of **B**, i.e. functions $e:X\to\operatorname{obj}\mathbf{B}$, where X is a set; a 1-cell $r:(X,e)\to(Y,e')$ is an $X\times Y$ matrix $(r_{xy})_{x,y\in X\times Y}$ of 1-cells r_{xy} in **B**, with $r_{xy}:e(x)\to e'(y)$. Composition of 1-cells $X\xrightarrow{r}Y\xrightarrow{s}Z$ is still matrix multiplication, i.e. $(sr)_{xz}=\sum_{y\in Y}s_{yz}\cdot r_{xy}$, and 2-cells are again defined ‘pointwise’. With this definition **Matr(B)** is a bicategory with local small coproducts. Notice also that if **B** is a one-object bicategory, that is a monoidal category, this definition agrees with the previous one.

Observe that matrices are exactly *discrete* **B**-categories, that is categories X for which $X(x,y)$ is 0_{xy} if $x \neq y$ and is 1_{ex} if $x=y$.

Proposition 4.8 (Idempotence theorem for **Matr(B)**). *If* **B** *is a bicategory with small local coproducts stable under composition, the bicategory* **Matr(B)** *admits collages of small matrices and hence* **Matr(B)** *is biequivalent to* **Matr(Matr(B))**.

Proof. Since composition of 1-cells does not involve coequalizers, we follow once more the line of Proposition 2.2, even if **B** is not distributive. The collage of a matrix (X,e) on **Matr(B)**, that is a function $e: X \to \mathrm{obj}\,$**Matr(B)**, is the object $W = \coprod_{x \in X} e(x)$ of **Matr(B)**. □

Corollary 4.9 (Characterization theorem for **Matr(B)**). *Let* **B** *be a bicategory with small local coproducts stable under composition. Then the following conditions are equivalent:*

(i) **B** *admits collages of small matrices,*

(ii) **B** *is biequivalent to* **Matr(W)** *for some bicategory* **W** *with small local stable coproducts.*

Notice that the left universal property of the collage of a matrix is exactly the universal property of the bicoproduct of the corresponding families of objects.

Corollary 4.10. *A bicategory* **B** *with stable local small coproducts is biequivalent to* **Matr(W)** *for some bicategory* **W** *with stable local small coproducts iff* **B** *has small bicoproducts.*

Remark 4.11. If **B** is a distributive bicategory, then **Matr(B)** is still distributive (just recall that 2-cells are defined ‘pointwise’). Therefore we can perform the **Mon( )** construction and observe, just recalling the definitions, that **Mon(Matr(B))** is biequivalent to **B-mod**.

Referring once more to Proposition 2.5, we now give a simple proof of a theorem proved by Street [16].

Proposition 4.12. *If a distributive bicategory* **B** *has small bicoproducts and is Kleisli-complete, then it admits collages.*

Proof. We have to show that **B** $\simeq$ **B-mod**. By Corollaries 4.3 and 4.10 we have **B** $\simeq$ **Mon(B)** and **B** $\simeq$ **Matr(B)**, hence **B** $\simeq$ **Mon(Matr(B))** $\simeq$ **B-mod**. □

Remark 4.13. Notice that the constructions **Mon**( ) and **Matr**( ) are not interchangeable: **Mon(Matr(B))** is different from **Matr(Mon(B))**, which is not even always Kleisli-complete, as is shown by the following example. Let $\mathbf{B}=\mathbf{2}$. Then **Mon(Matr(2))** is **Ord**(S), whereas **Matr(Mon(2))** is **Matr(2)**, that is, it is **Rel**(S), which is not Kleisli-complete.

5. Density

Let **B** be a distributive bicategory. A full subbicategory **W** of **B** is *dense* if every object of **B** is the collage of a small **W**-category. Notice that from the definition and from Remark 2.3 it follows that **W** *is dense in* **W-mod**.

Remark 5.1. Let **C** and **B** be distributive bicategories. A homomorphism of bicategories $F:\mathbf{C}\to\mathbf{B}$ induces a homomorphism $F^*:\mathbf{C\text{-}mod}\to\mathbf{B\text{-}mod}$ defined as follows: if X is a **C**-category, F^*X is the **B**-category with the same objects as X and $F^*X(x,y)=F(X(x,y))$. Further, if $R:X\to Y$ is a **C**-module, the **B**-module $F^*R:F^*X\to F^*Y$ is given by $F^*R(x,z)=F(R(x,z))$, where $x\in X$ and $z\in Y$.

Let us denote by $J:\mathbf{W}\to\mathbf{B}$ the inclusion of a full subbicategory **W** in **B**.

Proposition 5.2. *A full subbicategory* **W** *of* **B** *is dense iff the embedding homomorphism* $(\;)\hat{}:\mathbf{B}\to\mathbf{B\text{-}mod}$ *factors through* $J^*:\mathbf{W\text{-}mod}\to\mathbf{B\text{-}mod}$ (*up to equivalence*).

Proof. Let us construct a homomorphism $G:\mathbf{B}\to\mathbf{W\text{-}mod}$ which assigns to each b in **B** a *chosen* **W**-category X such that $\mathrm{coll}\,X=b$ and to each arrow $p:b\to b'$ the module $P:X\,\mathrel{+\!\!\!\to}\,Y$ (Y is chosen such that $\mathrm{coll}\;Y=b'$) obtained from $\hat{p}:\hat{b}\,\mathrel{+\!\!\!\to}\,\hat{b}'$ and the equivalences $\hat{b}\underset{S}{\overset{R}{\rightleftarrows}}X$, $\hat{b}'\underset{S'}{\overset{R'}{\rightleftarrows}}Y$ as $P=R'pS$. Then $J^*G\simeq(\;)\hat{}$ in **HOM(B, B-mod)**. The converse is obvious. $\square$

Proposition 5.3. *A full subbicategory* **W** *of* **B** *is dense iff the inclusion homomorphism* $J^*:\mathbf{W\text{-}mod}\to\mathbf{B\text{-}mod}$ *is a biequivalence.*

Proof. Suppose **W** is dense in **B**. The homomorphism J^* induces, for each pair of **W**-categories X,X', an obvious isomorphism of categories $\mathbf{W\text{-}mod}(X,X')\simeq\mathbf{B\text{-}mod}(X,X')$. To show that every object of **B-mod** is equivalent in **B-mod** to a **W**-category, notice first that an equivalence between two homomorphisms F and G in **HOM(W, B)** lifts to an equivalence between the induced homomorphisms F^*,G^* in **HOM(W-mod, B-mod)**. Hence the equivalence $J^*G\simeq(\;)\hat{}$ of Proposition 5.2 lifts to an equivalence $J^{**}G^*\simeq(\;)\hat{}:\mathbf{B\text{-}mod}\to(\mathbf{B\text{-}mod})\text{-}\mathbf{mod}$. Therefore, for any object X of **B-mod** we have $\hat{X}\simeq G^*X$ in **(B-mod)-mod** and, since homomorphisms preserve equivalences, $X\simeq\mathrm{coll}\,\hat{X}\simeq\mathrm{coll}\;G^*X$. But **W-mod** has collages, hence $\mathrm{coll}\;G^*X$ is an object of **W-mod**. Conversely, if J^* is a biequivalence, for every object of the form $\hat{b}$ (where b is an object of **B**) there exists a **W**-category Y such that $\hat{b} \simeq Y$ in **B-mod**. □

We are now able to produce a 'size sensitive' version of the characterization theorem.

Proposition 5.4 (Street [16]). *A distributive bicategory* **B** *is biequivalent to* **W-mod**, *for a* small *distributive bicategory* **W**, *iff* **B** *admits collages and there exists a* small *dense subbicategory of* **B**.

Proof. Let **B** have collages and a dense subbicategory **W**. Then we have by Proposition 5.3 that $\mathrm{coll}(\,)\cdot J^*: \mathbf{W\text{-}mod} \to \mathbf{B}$ is a biequivalence. Conversely, if **B** is biequivalent to **W-mod** (for a small **W**) just recall that **W-mod** has collages and that **W** is dense in **W-mod**. □

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